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The Replica-Symmetric Solution without Replica Trick for the Hopfield Model

L. Pastur,[1] M. Shcherbina,[1] and B. Tirozzi[2]

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We derive the saddle-point equations for the order parameters of the Hopfield model in the case of replica symmetry without using the replica trick, but assuming that the Edwards–Anderson parameter is a self-averaging quantity.

KEY WORDS: Replica symmetry; self-averageness; saddle-point equations.

1. INTRODUCTION

The Hopfield model is one of the most widely used models of the theory of disordered systems. It is defined by the usual spin Hamiltonian H_N (say, with the Ising spins $S_i = \pm 1$, $i = 1,..., N$) with the interaction of the form

$$J_{ij} = N^{-1} \sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu \tag{1.1}$$

where ξ_i^μ are independent identically distributed random variables. In other words, the interaction is the sum of p modes $\xi^\mu = \{\xi_1^\mu,..., \xi_N^\mu\}$.

This model with N-independent p was proposed in refs. 11–13 as a simple model of disordered spin systems, spin glasses in particular. Later the same model was successfully used[8,9] as a model of associative memory, where the random vectors ξ^μ with $\xi_i^\mu = \pm 1$ describe the learned memory patterns.

It is evident that the model defined by (1.1) is a model of the mean-field type. This fact allows for the rather complete analysis of the model for finite p or for $p \ll N$ (say $p/N \to 0$ for $N \to \infty$[6]). However, according to a

[1] Institute for Low Temperature Physics, Kharkov, 310164, Ukraine.

[2] Department of Mathematics, University of Rome "La Sapienza," 00185 Rome, Italy.

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widely accepted point of view (see, e.g., refs. 2 and 14), the Hopfield model with a finite p does not reflect many important properties of spin glasses and neural networks and the more realistic model corresponds to a macroscopically large number of patterns, i.e., $p/N = \alpha > 0$ for $p \to \infty$, $N \to \infty$. In particular it is natural to expect that if $\alpha \gg 1$ the Hopfield model is thermodynamically equivalent to the Sherrington–Kirkpatrick (SK) model in which $\{J_{ij}\}$ is a family of independent (except for the symmetry condition $J_{ij} = J_{ji}$) Gaussian random variables.

Both models have been extensively studied in the physical literature and many interesting and important results obtained. However, the main technical tool of the majority of these findings is the so-called "replica trick," which was invented in order to overcome the fundamental technical difficulty of the theoretical physics of quenched disordered systems. Namely, since in these systems the self-averaging (s.a.) property (i.e., nonrandomness in the macroscopic limit) holds for the free energy $f_N = -(\beta N)^{-1} \log Z_N$ but not for the partition function Z_N, the averaging procedure $E\{\cdots\}$ with respect to the random parameters (interactions J_{ij}, external fields h_i, etc.) has to be applied to f_N, but not to Z_N. The former procedure is as a rule very difficult to perform and the latter procedure is rather simple even if it is applied to Z_N^n for $n = 1, 2,...$. The replica trick reduces the former procedure to the latter one by using the elementary identity $\ln Z = \lim_{n \to 0}[(Z_N^n - 1)/n]$ and a rather subtle continuation of the sequence $E\{Z_N^n\}_{n=1,2,...}$ to the continuously varying and tending to zero values of n. It is just the nonuniqueness of this limit[16] that disallows justifying the choice of the SK free energy.

The replica trick, being a most popular and pragmatically rather efficient technical tool in the theoretical physics of disordered systems, is rather poorly understood mathematically. In ref. 4 it was shown that the simplest so-called "replica-symmetric" solution of the SK model is a rigorous consequence of the s.a. property of the spin-glass order parameter, known as the Edwards–Anderson parameter. Thus, at least for the part of the phase diagram of the SK model where the replica symmetry is unbroken, the replica-symmetric solution is the rigorous consequence of the s.a. property, which is easier to understand and which is valid in many models with a short-range interaction. On the other hand, for the part of the phase diagram of the SK model lying below the de Almeida–Thouless line, where the replica symmetry is broken, the rigorous result of ref. 4 is in agreement with the Parisi theory predicting the absence of the s.a. property of the Edwards–Anderson order parameter.

The aim of this paper is to prove rigorously the analogous results for the Hopfield model. The replica-symmetric solution in this model was found by Amit *et al.*,[1] who derived the system of self-consistent equations for the set of the order parameters of the model, known as the saddle-point equations. Our strategy is in essence the same as in ref. 4 and is based on the careful comparison of certain thermodynamic quantities corresponding to the systems of $N-1$ and N spins. However, the technical side of our proofs in this paper is more complicated than in the case of the SK model. This is not too surprising since, as was mentioned above, the SK model can be regarded as the limiting case of the Hopfield model.[17]

The general idea is to define the "interpolating" Hamiltonian $H(\tau, \boldsymbol{\theta})$ of the system of $N-1$ spins $S_2,..., S_N$ with certain parameters τ and $\boldsymbol{\theta}$ such that for $\tau = S_1$ and certain $\boldsymbol{\theta}$ it coincides with the Hamiltonian H_N of the Hopfield model of N spins defined by (1.1) and for $\tau = 0$ it coincides with the Hopfield Hamiltonian H_{N-1} of $N-1$ spins $S_2,..., S_N$. Then we introduce the relative partition function

$$u(\tau) = \frac{\operatorname{Tr} e^{-\beta H(\tau, \boldsymbol{\theta})}}{\operatorname{Tr} e^{-\beta H(0, \boldsymbol{0})}} \tag{1.2}$$

and study its Taylor expansion with respect to τ up to the second order. The fact that the coefficients of this Taylor expansion have a suitable form and are s.a. allows us to derive the equations for the order parameters q, r, m^μ which are identical to those of Amit *et al.* in the case of replica symmetry. We remark that we have to add more terms than the usual ones in the Hamiltonian (the ε_1 and ε_2 terms of Definition 1, Section 2) in order to prove the s.a. properties of certain quantities and the convergence to zero of the remainder of the Taylor expansion of $u(\tau)$.

The plan of the paper is the following. In Section 2 we give the main definitions and discuss them, state the main lemmas on the properties of s.a. and of the Taylor expansion of the function $u(\tau)$, and, using these results, derive the saddle-point equations. In Section 3 we give the proofs of the lemmas.

2. DERIVATION OF THE SADDLE-POINT EQUATIONS

Let the patterns be defined as a set of p random vectors $\{\xi_1^\mu,..., \xi_N^\mu\}$, $\mu = 1,..., p$, whose components are all independent and identically distributed with values $\xi_i^\mu = \pm 1$ and zero mean. Let also

$$\xi = \{\xi_i^\mu\}_{i=1,...,N}^{\mu=1,...,p}$$

be the set of all these variables and $S_i = \pm 1$ be the neuronal activities (=spin variables).

Definition 1:

$$t^\mu = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \xi_i^\mu S_i$$

$$t_1^\mu = \frac{1}{\sqrt{N}} \sum_{i=2}^{N} \xi_i^\mu S_i \tag{2.1}$$

Definition 2:

$$H_0 = -\frac{1}{2N} \sum_{\mu=k+1}^{p} \sum_{i,j=1}^{N} \xi_i^\mu \xi_j^\mu S_i S_j - \varepsilon_1 \sum_{i=1}^{N} h_i S_i - \varepsilon_2 \sum_{\mu=k+1}^{p} \gamma^\mu t^\mu \tag{2.2}$$

$$H(k) = -\frac{1}{2N} \sum_{\mu=1}^{k} \sum_{i,j=1}^{N} \xi_i^\mu \xi_j^\mu S_i S_j - \sum_{\mu=1}^{k} \varepsilon^\mu \sum_{i=1}^{N} \xi_i^\mu S_i$$

$$H = H_0 + H(k) \tag{2.3}$$

where $\{h_i\}$, $\{\gamma^\mu\}$ are independent Gaussian random variables with zero mean and variance 1, $E\{\cdots\}$ is the expectation w.r.t. ξ, $\{h_i\}$, and $\{\gamma^\mu\}$, and $\langle \cdot \rangle$ is the expectation w.r.t. the finite-volume Gibbs distribution generated by H.

H is the starting Hamiltonian for our analysis; it differs from the canonical form of the Hopfield model[1,2] by some auxiliary fields given by the terms containing ε_1, ε_2, ε^μ ($\mu = 1,..., k$). These fields are introduced following the general strategy of statistical mechanics already applied in the case of the ferromagnetic Ising model in order to study the phase transition (more exactly, in order to find the spontaneous magnetization in the Ising model). The presence of these fields makes the canonical Gibbs averages different from zero for any finite N. Then, after the thermodynamic limit is done, we send, as in the Ising model, the intensivity of the auxiliary fields to zero and study the limiting equations. Moreover, let us note that the terms containing ε_1, ε_2, and ε^μ will be used below for proving the self-averaging property of r and m^μ in the limit $N \to \infty$ and for obtaining a suitable expression for the coefficient of the second-order term in the Taylor expansion of the interpolating function $u(\tau)$ specified by (1.1). The ε_1 term is also used in order to find an upper bound for the remainder of the Taylor expansion, which vanishes as $N \to \infty$. This is clear from the proofs shown at the end of Section 3.

Definition 3:

$$
\begin{aligned}
r &= \frac{1}{p} \sum_{\mu=k+1}^{p} \langle t^{\mu} \rangle^2 \\
q &= \frac{1}{N} \sum_{i=1}^{N} \langle S_i \rangle^2 \qquad (2.4) \\
m^{\mu} &= \frac{1}{N} \sum_{i=1}^{N} \xi_i^{\mu} \langle S_i \rangle
\end{aligned}
$$

m^{μ} is known as the overlap with a given pattern configuration, q is the Edward–Anderson parameter, and r is the parameter which takes into account the influence of the patterns whose overlap vanishes in the limit $N \to \infty$.[1] This corresponds also to the special splitting that we introduced in the Hamiltonian (2.3) where the patterns which condense [i.e., $(m^{\mu})^2 > 0$ for $N \to \infty$] are separated from the ones which have a "collective" influence on the memorization process as a Gaussian noise. We cannot consider here the question of which measure the finite-N Gibbs distribution generates in the thermodynamic limit, because it is an unsolvable problem due to the fact that the random couplings have no decrease property for large distance among the points.

Definition 4:

$$
\begin{aligned}
H_0^1 &= -\frac{1}{2N} \sum_{\mu=k+1}^{p} \sum_{i,j=2}^{N} \xi_i^{\mu} \xi_j^{\mu} S_i S_j - \varepsilon_1 \sum_{i=2}^{N} h_i S_i - \varepsilon_2 \sum_{\mu=k+1}^{p} \gamma^{\mu} t_1^{\mu} \\
H^1(k) &= -\frac{1}{2N} \sum_{\mu=1}^{k} \sum_{i,j=2}^{N} \xi_i^{\mu} \xi_j^{\mu} S_i S_j - \sum_{\mu=1}^{k} \varepsilon^{\mu} \sum_{i=2}^{N} \xi_i^{\mu} S_i \\
H^1 &= H_0^1 + H^1(k) \qquad (2.5) \\
\tilde{h}_1 &= \sum_{\mu=1}^{k} \varepsilon^{\mu} \xi_1^{\mu} + \varepsilon_1 h_1 + \frac{\varepsilon_2}{\sqrt{N}} \sum_{\mu=k+1}^{p} \gamma^{\mu} \xi_1^{\mu} \\
H(\tau, \boldsymbol{\theta}) &= H^1 - \frac{\tau}{\sqrt{N}} \sum_{\mu=1}^{p} \xi_1^{\mu} \theta^{\mu} t_1^{\mu}
\end{aligned}
$$

where $\boldsymbol{\theta} = \{1, \ldots, 1, \theta^1, \theta^2, \theta^3, \theta^4, 1, \ldots, 1\}$ and the number of 1 before the θ variables is k; $\mathbf{1}$ will be the special configuration of θ given by $\mathbf{1} = \{1, \ldots, 1, 1, 1, 1, 1\}$; $\langle \cdots \rangle_{\tau, \boldsymbol{\theta}}$ will be the expectation w.r.t. the Gibbs measure generated by $H(\tau, \boldsymbol{\theta})$.

For $\tau = S_1$ and $\boldsymbol{\theta} = \mathbf{1}$, $H(\tau, \boldsymbol{\theta}) - \tilde{h}_1 \tau$ coincides with the Hamiltonian (2.3). It is introduced in the spirit of the cavity method[2] as an interpolating Hamiltonian between the system with $N-1$ spins and the one with N spins. The parameters θ^1, θ^2, θ^3, θ^4 are introduced for the proof of Lemma 2.2 below: they will be used in the estimate of the remainder of the respective Taylor expansion and afterward will be set equal to **1** to reconstruct the Hamiltonian of the N neurons. In fact the term in (2.5) which multiplies τ represents the interaction between the neuron located at the lattice site 1 with those of the $N-1$ system.

Definition 5:

$$u(\tau)=\ln\frac{\operatorname{Tr} e^{-\beta H(\tau,1)}}{\operatorname{Tr} e^{-\beta H(0,1)}}$$

The function $u(\tau)$ is the object to be studied in this paper because it is the main tool which allows us to derive Eqs. (2.6)–(2.8) below for the order parameters of the model.

Apart from the normalizing constant, $\exp[u(\tau)]$ is the partition function of the system of $(S_2,..., S_N)$ neurons with the external field introduced in (2.5). The key point of our method is to compute $\langle S_1\rangle$ by means of $\exp[u(\tau)]$ and to obtain a simple expression for $u(\tau)$ by using the Taylor formula of Lemma 2.2.

Definition 6. A random variable φ is self-averaging (s.a.) for $N\to\infty$ if

$$E\{\varphi^2\}-E\{\varphi\}^2\to 0$$

In this paper the limit $N\to\infty$ is done sending also $p\to\infty$ with $\alpha=p/N$ fixed. α is known as the "capacity" of the network.

Definition 7 (free energy):

$$f=\lim_{N\to\infty}-\frac{1}{N}\ln\operatorname{Tr} e^{-\beta H}$$

Our result follows from two main lemmas:

Lemma 2.1:

(a) m^μ is s.a.

(b) If in some range of the parameters β, ε_1, ε_2, ε^μ, α the parameter q is s.a., then also r is s.a.

Lemma 2.2. If in some range of the parameters β, ε_1, ε_2, ε^μ, α the parameter q is s.a., then

$$u(\tau)=\beta\tau(\alpha r)^{1/2}\,v+\beta\tau\sum_{\mu=1}^{k}\xi_1^\mu m^\mu+U\tau^2\beta^2/2+R_N(\tau)$$

where $E\{R_N^2(\tau)\} \to 0$ as $N \to \infty$,

$$U = \frac{1}{N} \sum_{\mu=1}^{p} \langle (t^\mu - \langle t^\mu \rangle)^2 \rangle$$

and v is a Gaussian random variable with zero mean and variance 1.

The proofs of these two lemmas are rather technical and we postpone them to the next section. Here we prove the main result, assuming that these lemmas are true.

These lemmas allow us to start the derivation of the equations for m^μ, r, and q. Lemma 2.1 is divided into two parts. The first one does not depend on the s.a. property of q: it will be shown in Section 3 by taking the derivative of the free energy w.r.t. ε^μ. The second part is interesting by itself and not only for its applications: it shows in fact that there is a hierarchy in the s.a. property: once q is self-averaging, r is also s.a. In this way we get that the central limit theorem holds for the sum of random variables which appears as the first term in the formula for $u(\tau)$ in Lemma 2.2. The main points to show in Section 3 are the proof that U has the form given in Lemma 2.2 and that the remainder $R_N(\tau)$ goes to zero in probability when $N \to \infty$.

Theorem 1. If for $\beta \in (\beta_0, \beta_0 + \delta)$, ε_1, ε_2, $\varepsilon^\mu \in (0, \delta)$, $\alpha \in (\alpha_0, \alpha_0 + \delta)$, the parameter q is s.a., then in the limits $N \to \infty$, $p \to \infty$, $p/N \to \alpha$, $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$ taken in the prescribed order, we have

$$m^\mu = E\left\{\int \frac{dv \exp(-v^2/2)}{(2\pi)^{1/2}} \xi_1^\mu \tanh \beta \left[(\alpha r)^{1/2} v + \sum_{\mu=1}^{k} (m^\mu + \varepsilon^\mu) \xi_1^\mu \right]\right\} \tag{2.6}$$

$$r = \frac{q}{(1 - \beta + \beta q)^2} \tag{2.7}$$

$$q = E\left\{\int \frac{dv \exp(-v^2/2)}{(2\pi)^{1/2}} \tanh^2 \beta \left[(\alpha r)^{1/2} v + \sum_{\mu=1}^{k} (m^\mu + \varepsilon^\mu) \xi^\mu \right]\right\} \tag{2.8}$$

Remark. The order of the limits in Theorem 1 cannot be interchanged because in the bound for $ER_N^2(\tau)$ we get a term $\varepsilon_2^{-4} o(1)$, $N \to \infty$.

Derivation of Formula (2.6). Since m^μ is s.a. ($\mu = 1,..., k$), then for $N \to \infty$

$$m^\mu = E\{\xi_1^\mu \langle S_1 \rangle\} = E\left\{\frac{1}{N} \sum_{i=1}^{N} \xi_i^\mu \langle S_i \rangle\right\} + o(1)$$

The expression for $\langle S_1 \rangle$ can be obtained from Definitions 4 and 5 and Lemma 2.2:

$$\langle S_1 \rangle = \frac{e^{u(1)+\beta\tilde{h}_1} - e^{u(-1)-\beta h_1}}{e^{u(1)+\beta\tilde{h}_1} + e^{u(-1)-\beta\tilde{h}_1}} = \tanh \beta \left[(\alpha r)^{1/2} v + \sum_{\mu=1}^{k} (m^\mu + \varepsilon^\mu) \xi^\mu \right] \quad (2.9)$$

Since, as we will see in the proof of Lemma 2.2, v is obtained as a sum of i.i.d. random variables $\{\xi_1^\mu\}_{\mu=1}^{p}$, then the expectation over ξ implies an average w.r.t. v whenever this variable appears. Applying this argument to (2.9), we get formula (2.6).

Derivation of Formula (2.8). According to (2.2)–(2.4) and the hypotheses of the theorem,

$$q = E\left\{\frac{1}{N}\sum_{i=1}^{N} \langle S_i \rangle^2\right\} + o(1) = E\{\langle S_1 \rangle^2\} + o(1), \qquad N \to \infty$$

Inserting (2.9) in the last equation, we obtain (2.8).

Derivation of (2.7). Let us start from the study of the quantity U defined in Lemma 2.2 above:

$$E\left\{\frac{1}{N}\sum_{\mu=1}^{p} \langle (t^\mu)^2 \rangle\right\} = E\left\{\frac{1}{N}\sum_{\mu=1}^{p} 1 + \frac{1}{\sqrt{N}}\sum_{\mu=1}^{p} \xi_1^\mu \langle S_1 t_1^\mu \rangle\right\} \quad (2.10)$$

Let φ be a function only of the spins $S_2,..., S_N$. We use the identity

$$\langle \varphi S_1 \rangle = \frac{\langle \varphi \rangle_{+1,1} e^{\{u(1)+\beta\tilde{h}_1\}} - \langle \varphi \rangle_{-1,1} e^{\{u(-1)-\beta\tilde{h}_1\}}}{e^{\{u(1)+\beta\tilde{h}_1\}} + e^{\{u(-1)-\beta\tilde{h}_1\}}}$$

where $\langle \cdot \rangle_{+1,1}$, $\langle \cdot \rangle_{-1,1}$ were specified in Definition 4. According to this identity and (2.10),

$$E\left\{\frac{1}{N}\sum_{\mu=1}^{p} \langle (t^\mu)^2 \rangle\right\}$$
$$= \alpha + \frac{1}{\beta} E\left\{\frac{(du/d\tau)|_{\tau=1} e^{\{u(1)+\beta\tilde{h}_1\}} - (du/d\tau)|_{\tau=-1} e^{\{u(-1)-\beta\tilde{h}_1\}}}{e^{\{u(1)+\beta\tilde{h}_1\}} + e^{\{u(-1)-\beta\tilde{h}_1\}}}\right\}$$

Now, using Lemma 2.2 and the Griffith lemma on the convergence of the first derivatives of convex functions [note that $u(\tau)$ is a convex function], one can obtain that in the limits $N \to \infty$, $p \to \infty$, $p/N \to \alpha$, $\varepsilon_{1,2} \to 0$ taken in the prescribed order,

$$E\left\{\frac{1}{N}\sum_{\mu=1}^{p}\langle(t^\mu)^2\rangle\right\}$$

$$=\alpha+E\left\{\frac{[(\alpha r)^{1/2}v+\sum_{\mu=1}^{k}\xi_1^\mu m^\mu+U\beta]\,e^C-[(\alpha r)^{1/2}v+\sum_{\mu=1}^{k}\xi_1^\mu m^\mu-U\beta]\,e^{-C}}{e^C+e^{-C}}\right\}$$

$$=\alpha+E\left\{\left[(\alpha r)^{1/2}v+\sum_{\mu=1}^{k}\xi_1^\mu m^\mu\right]\tanh C\right\}+U\beta \tag{2.11}$$

where

$$C=\beta\left[(\alpha r)^{1/2}v+\sum_{\mu=1}^{k}(m^\mu+\varepsilon^\mu)\,\xi_1^\mu\right]$$

Using the same arguments and the identity

$$\langle\varphi\rangle=\frac{\langle\varphi\rangle_{+1,1}\,e^{\{u(1)+\beta\tilde{h}_1\}}+\langle\varphi\rangle_{-1,1}\,e^{\{u(-1)-\beta\tilde{h}_1\}}}{e^{\{u(1)+\beta\tilde{h}_1\}}+e^{\{u(-1)-\beta\tilde{h}_1\}}}$$

we obtain

$$E\left\{\frac{1}{N}\sum_{\mu=1}^{p}\langle t^\mu\rangle^2\right\}$$

$$=E\left\{\frac{1}{N}\sum_{\mu=1}^{p}\langle S_1\rangle^2+\frac{1}{\sqrt{N}}\sum_{\mu=1}^{p}\xi_1^\mu\langle S_1\rangle\langle t_1^\mu\rangle\right\}$$

$$=\alpha q+\frac{1}{\beta}E\left\{\langle S_1\rangle\frac{(du/d\tau)|_{\tau=1}\,e^{\{u(1)+\beta\tilde{h}_1\}}+(du/d\tau)|_{\tau=-1}\,e^{\{u(-1)-\beta\tilde{h}_1\}}}{e^{\{u(1)+\beta\tilde{h}_1\}}+e^{\{u(-1)-\beta\tilde{h}_1\}}}\right\}$$

$$\to\alpha q+E\left\{\langle S_1\rangle\left[(\alpha r)^{1/2}v+\sum_{\mu=1}^{k}m^\mu\xi_1^\mu\right]\right\}+\beta UE\{\langle S_1\rangle\tanh C\}$$

and using (2.9), we have for $\varepsilon_{1,2}\to 0$

$$E\left\{\frac{1}{N}\sum_{\mu=1}^{p}\langle t^\mu\rangle^2\right\}$$

$$=\alpha q+\beta Uq+E\left\{\left[(\alpha r)^{1/2}v+\sum_{\mu=1}^{k}m^\mu\xi_1^\mu\right]\tanh C\right\} \tag{2.12}$$

Now, subtracting (2.12) from (2.11), we find that

$$U = \alpha(1-q) + \beta U(1-q)$$

or

$$U = \frac{\alpha(1-q)}{1-\beta(1-q)} \tag{2.13}$$

In order to derive Eq. (2.7) now it suffices to take into account that

$$\begin{aligned} r &= \frac{1}{p} \sum_{\mu=k+1}^{p} \langle t^\mu \rangle^2 \\ &= \frac{1}{\alpha N} \sum_{\mu=1}^{p} \langle t^\mu \rangle^2 - \frac{1}{\alpha} \sum_{\mu=1}^{k} \{m^\mu\}^2 \\ &= \frac{1}{\alpha} \left[\alpha q + \beta U q + E \left\{ \left((\alpha r)^{1/2} v + \sum_{\mu=1}^{k} m^\mu \xi_1^\mu \right) \tanh C \right\} - \sum_{\mu=1}^{k} \{m^\mu\}^2 \right] \end{aligned}$$

Therefore Eq. (2.6) yields

$$\begin{aligned} r &= q + \frac{\beta}{\alpha} Uq + \frac{1}{\alpha} \int \frac{dv \exp(-v^2/2)}{(2\pi)^{1/2}} (r\alpha)^{1/2} v \tanh C \\ &= q + \frac{\beta}{\alpha} Uq + \beta r \left\{ \int \frac{dv \exp(-v^2/2)}{(2\pi)^{1/2}} \cosh^{-2} C \right. \\ &= q + \frac{\beta}{\alpha} Uq + \beta r(1-q) \end{aligned}$$

Inserting the expression of U from (2.13) in this formula, one gets (2.7).

3. PROOFS OF LEMMAS 2.1 AND 2.2

Everywhere in this section we will use the notation

$$\langle \dot{A} B \rangle_{\zeta,\theta} \equiv \langle (A - \langle A \rangle_{\zeta,\theta})\, B \rangle_{\zeta,\theta}$$

where $\langle \cdots \rangle_{\zeta,\theta}$ was introduced in Definition 4.

Before starting the proofs, we sketch the general strategy used in this section. The order of the lemmas is difficult to optimize and so we have chosen a scheme of proof which we want to explain before starting to go through it.

(a) The remainder $R_N(\tau)$ in Lemma 2.2 is decomposed into the sum of three terms:

$$R = R_1 + 2R_2 + R_3$$

In Lemmas 3.2–3.4 we demonstrate that these terms go to zero as $N \to \infty$. The result of Lemma 3.1 is used in Lemmas 3.3 and 3.4 in order to bound R_2 and R_3 by a quantity which goes to zero as $N \to \infty$. Lemma 3.1 is shown using a formula obtained by integration by parts with respect to h_i as in ref. 4 and by starting from s.a. properties of the derivative of the free energy w.r.t. ε_2. Since Lemma 3.1 is used many times we put it at the beginning of this section.

(b) The proof of Lemma 2.2 is a based on Taylor expansion of $u(\tau)$ and the s.a. property of r and m^μ. We also use Lemma 3.5 in order to express the quantity U introduced in Lemma 2.2 in terms of the usual Gibbs measure $\langle \cdot \rangle$ instead of the less comfortable one $\langle \cdot \rangle_{\zeta,1}$.

(c) The proof of these last three facts is shifted to the end of Section 3. First we prove Lemma 3.5: the proof of this lemma is reduced to the proof of the s.a. of r, which is given at the end of the section. Also in this case we start from the derivative of the free energy with respect to ε_2.

Lemma 3.1. If q is s.a. and $\dot{t}_1^\mu$ are defined as in (2.1), then

$$S(\zeta) \equiv E\left\{\frac{1}{N^2} \sum_{\mu,\nu=1}^{p} \langle \dot{t}_1^\mu \dot{t}_1^\nu \rangle_{\zeta,1}^2 \right\} \to 0 \tag{3.1}$$

as $N \to \infty$ and also

$$E\left\{\frac{1}{N^2} \sum_{\mu=1}^{p} \langle (\dot{t}_1^\mu)^2 \rangle_{\zeta,1}^2 \right\} < S(\zeta) \to 0$$

Proof. From ref. 4 we know that if q is s.a., then

$$E\left\{\frac{1}{N^2} \sum_{ij}^{N} \langle \dot{S}_i \dot{S}_j \rangle_{0,1}^2 \right\} \to 0 \tag{3.2}$$

as $N \to \infty$. Let us set $Q_{ij} = \langle \dot{S}_i \dot{S}_j \rangle_{0,1}$. Then for

$$J_{ij} \equiv \frac{1}{N} \sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu$$

we have, in view of (2.1) and (3.1),

$$S(0) = E\left\{\frac{1}{N^2}\sum_{\mu,\nu=1}^{p}\sum_{i_1,i_2,j_1,j_2=2}^{N}\frac{1}{N^2}\xi_{i_1}^{\mu}\xi_{j_1}^{\mu}\xi_{i_2}^{\nu}\xi_{j_2}^{\nu}\right.$$
$$\left.\times\langle\dot{S}_{i_1}\dot{S}_{i_2}\rangle_{0,1}\langle\dot{S}_{j_1}\dot{S}_{j_2}\rangle_{0,1}\right\}$$
$$= E\left\{\frac{1}{N^2}\sum_{i_1,i_2} J_{i_1,j_1}J_{i_2,j_2}Q_{i_1i_2}Q_{j_1,j_2}\right\}$$
$$= E\left\{\frac{1}{N^2}\operatorname{Tr} JQJQ\right\}$$
$$= E\left\{\frac{1}{N^2}\operatorname{Tr} J^{1/2}QJQJ^{1/2}\right\}$$
$$\leqslant \frac{1}{N^2}E\{\|J\|\operatorname{Tr} QJQ\}$$
$$\leqslant \frac{1}{N^2}E\{\|J\|^2\operatorname{Tr} Q^2\}$$
$$\leqslant \text{const}\,\frac{1}{N^2}E\{\operatorname{Tr} Q^2\}$$

Here we have used the result of ref. 6, which implies that

$$\operatorname{Prob}\{\|J\| > (1+\sqrt{\alpha})^2 + \varepsilon\} \leqslant e^{\{-M\varepsilon^{3/4}N^{2/3}\}} \tag{3.3}$$

where M does not depend on ε and N. Therefore everywhere below we will use the inequality $\|J\| \leqslant \text{const}$. Now using (3.2), we get that

$$S(0) \to 0, \qquad N \to \infty \tag{3.4}$$

But

$$S(\zeta) = E\left\{\frac{1}{N^2}\min_{c^\mu}\sum_{\mu,\nu=1}^{p}\langle(t_1^\mu - c^\mu)(t_1^\nu - c^\nu)\rangle_{\zeta,1}^2\right\}$$
$$\leqslant E\left\{\frac{1}{N^2}\sum_{\mu,\nu=1}^{p}\langle(t_1^\mu - \langle t_1^\mu\rangle_{0,1})(t_1^\nu - \langle t_1^\nu\rangle_{0,1})\rangle_{\zeta,1}^2\right\}$$
$$= E\left\{\frac{(1/N^2)\sum_{\mu,\nu=1}^{p}\langle \dot{t}_1^\mu\dot{t}_1^\nu \exp[(\zeta\sum_\mu \xi_1^\mu t_1^\mu\beta)/\sqrt{N}]\rangle_{0,1}^2}{\langle\exp[(\zeta\sum_\mu \xi_1^\mu t_1^\mu\beta)/\sqrt{N}]\rangle_{0,1}^2}\right\}$$
$$\leqslant E\left\{\frac{1}{N^2}\sum_{\mu,\nu=1}^{p}\left\langle \dot{t}_1^\mu\dot{t}_1^\nu\exp\left(\frac{\zeta\sum_\mu\xi_1^\mu t_1^\mu\beta}{\sqrt{N}}\right)\right\rangle_{0,1}^2\right.$$
$$\left.\times\left\langle\exp\left(-\frac{\zeta\sum_\mu\xi_1^\mu t_1^\mu\beta}{\sqrt{N}}\right)\right\rangle_{0,1}^2\right\} \tag{3.5}$$

Here we have used the Jensen inequality

$$\frac{1}{\langle A\rangle_{0,1}}\leqslant\langle A^{-1}\rangle_{0,1}$$

for the derivation of the last expression. Now since $\langle\cdot\rangle_{0,1}$ does not depend on $\{\xi_1^\mu\}$ we can average the r.h.s. of Eq. (3.5) over $\{\xi_1^\mu\}$ and then, using the property

$$\frac{1}{N}\sum_\mu (t_1^\mu)^2=\frac{1}{N}\sum_{i,j=2}^{N} J_{ij}S_iS_j\leqslant\|J\| \tag{3.6}$$

obtain that

$$\begin{aligned}S(\zeta)&\leqslant E\left\{\frac{1}{N^2}\sum_{\mu,\nu=1}^{p}\langle \dot{t}_1^\mu\dot{t}_1^\nu\rangle_{0,1}^2\, e^{16\beta^2\|J\|}\right\}\\&\leqslant E^{1/2}\left\{\frac{1}{N^2}\sum_{\mu,\nu=1}^{p}\langle\dot{t}_1^\mu\dot{t}_1^\nu\rangle_{0,1}^2\right\}E^{1/2}\{\|J\|^2\, e^{32\beta^2\|J\|}\}\\&\leqslant S^{1/2}(0)\cdot\text{const}\end{aligned}$$

where we have used (3.3) to bound $E^{1/2}\{\|J\|^2\, e^{32\beta^2\|J\|}\}$. Now on the basis of (3.4) one can obtain (3.1). Lemma 3.1 is proved.

Proof of Lemma 2.2. Using Definition 5, we get the following Taylor formula for $u(\tau)$:

$$\begin{aligned}u(\tau)&=u'(0)\,\tau+\int_0^\tau d\zeta\; u''(\zeta)(\tau-\zeta)\\&=\beta\tau\sum_{\mu=1}^{k}\frac{\xi_1^\mu}{\sqrt{N}}\langle t_1^\mu\rangle_{0,1}+\beta\tau\sum_{\mu=k+1}^{p}\frac{\xi_1^\mu}{\sqrt{N}}\langle t_1^\mu\rangle_{0,1}\\&\quad+\beta^2\int_0^\tau d\zeta\,(\tau-\zeta)\sum_{\mu,\nu=1}^{p}\frac{\xi_1^\mu\xi_1^\nu}{N}\langle\dot{t}_1^\mu\dot{t}_1^\nu\rangle_{\zeta,1}\end{aligned} \tag{3.7}$$

By Definitions 4 and 5, $\langle t_1^\mu\rangle_{0,1}$ is independent of ξ_1^μ and one can apply the central limit theorem and Lemma 2.1 to the second sum in thr r.h.s. of this formula, getting for $N\to\infty$ a Gaussian random variable v with zero mean and variance:

$$E\left\{\left(\sum_{\mu=k+1}^{p}\frac{\xi_1^\mu}{\sqrt{N}}\langle t_1^\mu\rangle_{0,1}\right)^2\right\}\to\alpha r\qquad\text{as}\quad N\to\infty$$

In addition, by Definitions 1 and 3 due to the s.a. of m^μ

$$\frac{1}{\sqrt{N}}\langle t_1^\mu\rangle_{0,1}=m^\mu+o(1),\qquad N\to\infty,\quad \mu=1,\dots,k$$

we obtain the first two terms in the expansion of Lemma 2.2 for $u(\tau)$.

Now we consider the last term in (3.7). We first study the sum with two different indexes:

$$R=\sum_{\mu,\nu=1,\mu\neq\nu}^{p}\frac{\xi_1^\mu\xi_1^\nu}{N}\langle \dot{t}_1^\mu\dot{t}_1^\nu\rangle_{\zeta,1}=R_1+2R_2+R_3 \tag{3.8}$$

where

$$R_1=\sum_{\mu,\nu=1,\mu\neq\nu}^{k}\frac{\xi_1^\mu\xi_1^\nu}{N}\langle \dot{t}_1^\mu\dot{t}_1^\nu\rangle_{\zeta,1}$$

$$R_2=\sum_{\mu=1}^{k}\sum_{\nu=k+1}^{p}\frac{\xi_1^\mu\xi_1^\nu}{N}\langle \dot{t}_1^\mu\dot{t}_1^\nu\rangle_{\zeta,1}$$

$$R_3=\sum_{\mu,\nu=k+1,\mu\neq\nu}^{p}\frac{\xi_1^\mu\xi_1^\nu}{N}\langle \dot{t}_1^\mu\dot{t}_1^\nu\rangle_{\zeta,1}$$

Lemma 3.2:

$$\lim_{N\to\infty}E\{|R_1|\}=0$$

Proof:

$$\begin{aligned}
E\{|R_1|\}&\leqslant E\left\{\sum_{\mu,\nu=1,\mu\neq\nu}^{k}\frac{1}{N}|\langle \dot{t}_1^\mu\dot{t}_1^\nu\rangle_{\zeta,1}|\right\}\\
&\leqslant E\left\{\sum_{\mu=1}^{k}\frac{k}{N}\langle(\dot{t}_1^\mu)^2\rangle_{\zeta,1}\right\}\\
&=E\left\{\frac{k}{\varepsilon_2 N}\sum_{\mu=1}^{k}\gamma^\mu\langle t_1^\mu\rangle_{\zeta,1}\right\}\\
&=E\left\{\frac{k}{\varepsilon_2 N^{3/2}}\sum_{i=2}^{N}\langle S_i\rangle_{\zeta,1}\sum_{\mu=1}^{k}\gamma^\mu\xi_i^\mu\right\}\\
&\leqslant\frac{k}{\varepsilon_2 N^{1/2}}E^{1/2}\left\{\sum_{i=2}^{N}\frac{\langle S_i\rangle^2}{N}\right\}E^{1/2}\left\{\frac{1}{N}\sum_{i=2}^{N}\sum_{\mu=1}^{k}(\gamma^\mu\xi_i^\mu)^2\right\}\\
&\leqslant\frac{k^{3/2}}{\varepsilon_2 N^{1/2}}
\end{aligned} \tag{3.9}$$

Here we have used the Schwarz inequality and the formula

$$E\{\gamma\varphi(\gamma)\} = E\{\varphi'(\gamma)\} \tag{3.10}$$

which is valid for any Gaussian random variable with zero mean and variance one. Lemma 3.2 is proven.

Lemma 3.3:

$$\lim_{N\to\infty} E\{|R_2|\} = 0$$

Proof. By the Schwarz inequality we have

$$E\{|R_2|\} \leqslant E^{1/2}\left\{\sum_{\mu,\nu=1}^{k} \frac{1}{N} |\langle i_1^\mu i_1^\nu\rangle_{\zeta,1}|\right\}$$
$$\times E^{1/2}\left\{\sum_{\nu=k+1}^{p} \frac{1}{N} \langle (i_1^\nu)^2\rangle_{\zeta,1} + |R_3|\right\} \tag{3.11}$$

But according to (3.9), the first factor in the r.h.s. of this inequality has zero limit as $N\to\infty$. In addition, according to (3.6) and (3.3), the sum in the second factor is bounded. Finally, as it will be shown in Lemma 3.4, we have that

$$\lim_{N\to\infty} E\{R_3^2\} = 0$$

Combining these facts with (3.1), we obtain the lemma.

Lemma 3.4:

$$\lim_{N\to\infty} E\{R_3^2\} = 0$$

Proof:

$$E\{R_3^2\} = E\left\{\sum_{\mu_1\neq\nu_1,\mu_2\neq\nu_2}^{p} \frac{\xi_1^{\mu_1}\xi_1^{\nu_1}\xi_1^{\mu_2}\xi_1^{\nu_2}}{N^2} \langle i_1^{\mu_1} i_1^{\nu_1}\rangle_{\zeta,1} \langle i_1^{\mu_2} i_1^{\nu_2}\rangle_{\zeta,1}\right\}$$
$$= E\left\{\sum_{\mu_1\neq\mu_2\neq\nu_1\neq\nu_2}^{p} \cdots\right\} + 2E\left\{\sum_{\mu_1=\mu_2,\nu_1\neq\nu_2}^{p} \cdots\right\} + 2E\left\{\sum_{\mu_1\neq\mu_2,\nu_1=\nu_2}^{p} \cdots\right\}$$
$$\equiv E\{\Sigma_1\} + 2E\{\Sigma_2\} + 2E\{\Sigma_3\}$$

Let us consider the first sum in the r.h.s. of the last equality. The other sums can be estimated analogously. Since the Hamiltonian is symmetric w.r.t. the indices of the patterns $\mu = 1,..., p$, we can consider the case $\mu_1 = 1+k$, $\mu_2 = 2+k$, $\mu_3 = 3+k$, $\mu_4 = 4+k$ and multiply the term by a factor $(p-k)(p-k-1)(p-k-2)(p-k-3) \leqslant N^4\alpha^4$. Thus we have to estimate

$$E\{\Sigma_1\} = E\{N^2 \xi_1^{k+1}\xi_1^{k+2}\xi_1^{k+3}\xi_1^{k+4}\langle i_1^{k+1} i_1^{k+2}\rangle_{\zeta,1}\langle i_1^{k+3} i_1^{k+4}\rangle_{\zeta,1}\} \quad (3.12)$$

It is useful to introduce the function

$$\psi(\mathbf{\theta}) = \langle i_1^{k+1} i_1^{k+2}\rangle_{\zeta,\mathbf{\theta}}\langle i_1^{k+3} i_1^{k+4}\rangle_{\zeta,\mathbf{\theta}}$$

Then

$$E\{\xi_1^{k+1}\xi_1^{k+2}\xi_1^{k+3}\xi_1^{k+4}\psi(1)\}$$
$$= E\left\{\int_0^1 d\theta^1\, d\theta^2\, d\theta^3\, d\theta^4\, \xi_1^{k+1}\xi_1^{k+2}\xi_1^{k+3}\xi_1^{k+4} \frac{\partial^4 \psi(\mathbf{\theta})}{\partial\theta^1\,\partial\theta^2\,\partial\theta^3\,\partial\theta^4}\right\} \quad (3.13)$$

All the terms in the r.h.s. of (3.13) which have at least one of the variables $\theta^i = 0$, $i = 1,..., 4$, give a zero contribution to the expectation because of the independence of the ξ_i^μ. From Definition 4 we have the identity

$$\frac{\partial}{\partial\theta^i}\langle\cdot\rangle_{\zeta,\mathbf{\theta}} = \frac{\zeta\xi_1^{i+k}}{\varepsilon_2\sqrt{N}}\frac{\partial}{\partial\gamma^{i+k}}\langle\cdot\rangle_{\zeta,\mathbf{\theta}}$$

for $i = 1,..., 4$, which together with (3.12) gives for (3.13)

$$E\{\Sigma_1\} = \frac{\zeta^4}{\varepsilon_2^4}\int_0^1 d\theta^1\, d\theta^2\, d\theta^3\, d\theta^4\, E\left\{\frac{\partial^4\psi(\mathbf{\theta})}{\partial\gamma^{k+1}\,\partial\gamma^{k+2}\,\partial\gamma^{k+3}\,\partial\gamma^{k+4}}\right\}$$

Using formula (3.10), we have

$$E\{\Sigma_1\} = \frac{\zeta^4}{\varepsilon_2^4}\int_0^1 d\theta^1\, d\theta^2\, d\theta^3\, d\theta^4$$
$$\times E\{\gamma^{k+1}\gamma^{k+2}\gamma^{k+3}\gamma^{k+4}\langle i_1^{k+1} i_1^{k+2}\rangle_{\zeta,\mathbf{\theta}}\langle i_1^{k+3} i_1^{k+4}\rangle_{\zeta,\mathbf{\theta}}\}$$
$$\leqslant \frac{\zeta^4}{\varepsilon_2^4} E^{1/2}\{(\gamma^{k+1}\gamma^{k+2}\gamma^{k+3}\gamma^{k+4})^2\}$$
$$\times \int_0^1 d\theta^1\, d\theta^2\, d\theta^3\, d\theta^4\, E^{1/2}\{\langle i_1^{k+1} i_1^{k+2}\rangle_{\zeta,\mathbf{\theta}}^2\langle i_1^{k+3} i_1^{k+4}\rangle_{\zeta,\mathbf{\theta}}^2\} \quad (3.14)$$

We will concentrate now on the estimation of the quantity

$$E\{\langle i_1^{k+1} i_1^{k+2}\rangle_{\zeta,\mathbf{\theta}}^2\langle i_1^{k+3} i_1^{k+4}\rangle_{\zeta,\mathbf{\theta}}^2\} = D$$

It is useful to rewrite the quantity D in the following form:

$$D = E\{\langle t_1^{k+1} - \langle t_1^{k+1}\rangle_{\zeta,1})(t_1^{k+2} - \langle t_1^{k+2}\rangle_{\zeta,1})\rangle_{\zeta,\theta}^2$$
$$\times \langle (t_1^{k+3} - \langle t_1^{k+3}\rangle_{\zeta,1})(t_1^{k+4} - \langle t_1^{k+4}\rangle_{\zeta,1})\rangle_{\zeta,\theta}^2 + \text{Diff}\}$$
$$= \Delta + E\{\text{Diff}\} \tag{3.15}$$

where Diff is defined by

$$\text{Diff} = \langle \dot{t}_1^{k+1}\dot{t}_1^{k+2}\rangle_{\zeta,\theta}^2 \langle \dot{t}_1^{k+3}\dot{t}_1^{k+4}\rangle_{\zeta,\theta}^2$$
$$- \langle (t_1^{k+1} - \langle t_1^{k+1}\rangle_{\zeta,1})(t_1^{k+2} - \langle t_1^{k+2}\rangle_{\zeta,1})\rangle_{\zeta,\theta}^2$$
$$\times \langle (t_1^{k+3} - \langle t_1^{k+3}\rangle_{\zeta,1})(t_1^{k+4} - \langle t_1^{k+4}\rangle_{\zeta,1})\rangle_{\zeta,\theta}^2 \tag{3.16}$$

We will start by estimating Δ from (3.15). We use the identity

$$\langle \varphi\rangle_{\zeta,\theta} = \frac{\langle \varphi \exp[(\beta\zeta/\sqrt{N}) \sum_{\alpha=1}^4 (\theta_\alpha - 1)\, \xi_1^{\alpha+k} t_1^{\alpha+k}]\rangle_{\zeta,1}}{\langle \exp[(\beta\zeta/\sqrt{N}) \sum_{\alpha=1}^4 (\theta_\alpha - 1)\, \xi_1^{\alpha+k} t_1^{\alpha+k}]\rangle_{\zeta,1}} \tag{3.17}$$

and the bound

$$\exp(-4\beta) \leqslant \exp\left[\frac{\beta\zeta}{\sqrt{N}} \sum_{\alpha=1}^{4} (\theta_\alpha - 1)\, \xi_1^{\alpha+k} t_1^{\alpha+k}\right] \leqslant \exp(4\beta) \tag{3.18}$$

which follows from the simple estimate

$$\left|\frac{1}{\sqrt{N}}\, t_1^\alpha\right| = \left|\frac{1}{N} \sum_{i=2}^{N} \xi_i^\alpha S_i\right| \leqslant 1$$

Combining (3.17), (3.18), we have

$$\Delta \leqslant \text{const} \cdot E\left\{\left\langle \dot{t}_1^{k+1}\dot{t}_1^{k+2} \exp\left[\frac{\beta\zeta}{\sqrt{N}} \sum_{\alpha=1}^{4} (\theta_\alpha - 1)\, \xi_1^{\alpha+k} t_1^{\alpha+k}\right]\right\rangle_{\zeta,1}^2\right.$$
$$\left.\times \left\langle \dot{t}_1^{k+3}\dot{t}_1^{k+4} \exp\left[\frac{\beta\zeta}{\sqrt{N}} \sum_{\alpha=1}^{4} (\theta_\alpha - 1)\, \xi_1^{\alpha+k} t_1^{\alpha+k}\right]\right\rangle_{\zeta,1}^2\right\}$$

In the second factor of the r.h.s. of this inequality we use the bound (3.18), while in the first one we use the elementary inequality

$$|e^x - 1| \leqslant |x|\, e^{|x|}$$

and again the bound (3.18). We get

$$\Delta \leqslant E\{\langle \dot{t}_1^{k+1}\dot{t}_1^{k+2}\rangle_{\zeta,1}^2 \langle |\dot{t}_1^{k+3}|\cdot|\dot{t}_1^{k+4}|\rangle_{\zeta,1}^2\}$$
$$+\text{const}\cdot E\left\{\sum_{\alpha=1}^{4}\left\langle |\dot{t}_1^{k+1}|\cdot|\dot{t}_1^{k+2}|\frac{|t_1^{k+\alpha}|}{\sqrt{N}}\right\rangle_{\zeta,1}^2 \langle |\dot{t}_1^{k+3}|\cdot|\dot{t}_1^{k+4}|\rangle_{\zeta,1}^2\right\}$$
$$\equiv D_1+D_2 \tag{3.19}$$

Given the symmetry w.r.t. the indexes $\mu\geqslant k+1$ of the expectation $E\{\langle\cdot\rangle_{\zeta,1}\}$, we can estimate D_1 as follows:

$$D_1\leqslant E\left\{\frac{1}{N^4}\sum_{\mu_1,\ldots,\mu_4}\langle \dot{t}_1^{\mu_1}\dot{t}_1^{\mu_2}\rangle_{\zeta,1}^2\langle|\dot{t}_1^{\mu_3}|\cdot|\dot{t}_1^{\mu_4}|\rangle_{\zeta,1}^2\right\}$$

But according to the Schwarz inequality and the relations (3.3) and (3.6), which hold also for $\dot{t}_1^{\mu}$, we have

$$\frac{1}{N^2}\sum_{\mu_3,\mu_4}\langle|\dot{t}_1^{\mu_3}|\cdot|\dot{t}_1^{\mu_4}|\rangle_{\zeta,1}^2\leqslant\left[\frac{1}{N}\sum_{\mu}\langle(\dot{t}_1^{\mu})^2\rangle_{\zeta,1}\right]^2\leqslant 2\,\|J\|^2\leqslant\text{const}$$

Therefore on the basis of Lemma 3.1

$$D_1\leqslant\text{const}\cdot E\left\{\frac{1}{N^2}\sum_{\mu_1,\mu_2}\langle\dot{t}_1^{\mu_1}\dot{t}_1^{\mu_2}\rangle_{\zeta,1}^2\right\}\to 0$$

as $N\to\infty$.

Now let us find an estimate for the first term in the sum over α of (3.19); the other terms can be estimated similarly. First of all, using the symmetry of $E\{\langle\cdot\rangle_{\zeta,1}\}$ w.r.t. μ_i, we substitute one term with the sum over μ_1, μ_2, μ_3, μ_4 and then estimate the sum over μ_3, μ_4 as we have done for the second term by using (3.6) and (3.3). We observe also that we can interchange the factor $t_1^{k+\alpha}$ with $\dot{t}_1^{k+\alpha}$ in the definition (2.5) of $H(\tau,\boldsymbol{\theta})$ and the above derivation still holds since this change corresponds to a subtraction of a constant in the Hamiltonian (2.5). Thus

$$E\left\{\frac{1}{N^2}\sum_{\mu_1,\mu_2}\left\langle|\dot{t}_1^{\mu_1}|\cdot|\dot{t}_1^{\mu_2}|\frac{|\dot{t}_1^{\mu_1}|}{\sqrt{N}}\right\rangle^2\right\}$$
$$\leqslant E\left\{\frac{1}{N^2}\sum_{\mu_1}\langle(\dot{t}_1^{\mu_1})^2\rangle\,\|J\|\right\}$$
$$\leqslant E^{1/2}\{\|J\|^2\}\,E^{1/2}\left\{\frac{1}{N^4}\sum_{\mu_1,\mu_2}\langle(\dot{t}_1^{\mu_1})^2\rangle\langle(\dot{t}_1^{\mu_2})^2\rangle\right\}$$
$$\leqslant E^{1/2}\{\|J\|^2\}\,E^{1/2}\left\{\frac{1}{N^2}\sum_{\mu}\langle(\dot{t}_1^{\mu})^2\rangle^2\right\}$$

Here we have used (3.6) to estimate $(1/N)\sum_\mu (\dot{t}_1^\mu)^2$ and the Schwarz inequality in the form

$$\left(\int dx\, \rho f\right)^2 \leqslant \int dx\, \rho f^2, \qquad \int \rho\, dx = 1$$

Now since Lemma 3.1 implies that the r.h.s. of the last inequality has zero limit as $N\to\infty$, we have proved that $\Delta\to 0$ as $N\to\infty$. Let us now consider the expression (3.16). It has the form

$$E\{A^2 - B^2\} \leqslant E^{1/2}\{(A-B)^2\}\, E^{1/2}\{(A+B)^2\}$$

It is easy to bound the factor $E^{1/2}\{(A+B)^2\}$ by a constant as in the case of D_1 or D_2 in (3.19). Therefore we have to estimate

$$\begin{aligned} E\{(A-B)^2\} = E\{[&\langle \dot{t}_1^{k+1}\dot{t}_1^{k+2}\rangle_{\zeta,\boldsymbol{\theta}} \langle \dot{t}_1^{k+3}\dot{t}_1^{k+4}\rangle_{\zeta,\boldsymbol{\theta}} \\ &- \langle (t_1^{k+1} - \langle t_1^{k+1}\rangle_{\zeta,\mathbf{1}})(t_1^{k+2} - \langle t_1^{k+2}\rangle_{\zeta,\mathbf{1}})\rangle_{\zeta,\boldsymbol{\theta}} \\ &\times \langle (t_1^{k+3} - \langle t_1^{k+3}\rangle_{\zeta,\mathbf{1}})(t_1^{k+4} - \langle t_1^{k+4}\rangle_{\zeta,\mathbf{1}})\rangle_{\zeta,\boldsymbol{\theta}}]^2\} \\ \leqslant E\{(&\langle t_1^{k+1}\rangle_{\zeta,\boldsymbol{\theta}} - \langle t_1^{k+1}\rangle_{\zeta,\mathbf{1}})^2\}\cdot \text{const} \end{aligned}$$

Here we have used once more the symmetry of $E\{\langle\cdot\rangle\}_{\zeta,\mathbf{1}}$ w.r.t. $\mu\geqslant k+1$ and the bound (3.6) with the estimate (3.3).

Thus

$$\begin{aligned} E\{\text{Diff}\} &\leqslant E^{1/2}\{(\langle t_1^{k+1}\rangle_{\zeta,\boldsymbol{\theta}} - \langle t_1^{k+1}\rangle_{\zeta,\mathbf{1}})^2\} \\ &= \text{const}\cdot E^{1/2}\left\{\left(\int_0^1 d\lambda\, \frac{d}{d\lambda}\langle t_1^{k+1}\rangle_{\zeta,\boldsymbol{\theta}(\lambda)}\right)^2\right\} = C \end{aligned}$$

where $\boldsymbol{\theta}(\lambda)$ is the vector defined as $\boldsymbol{\theta}$ (see Definition 4), but with $\theta^i(\lambda) = 1 + \lambda(\theta^i - 1)$ for $i = 1,..., 4$, which interpolates between the vectors $\boldsymbol{\theta}$ and $\mathbf{1}$. Using this notation, we obtain the bounds

$$\begin{aligned} C &\leqslant E^{1/2}\left\{\left[\int_0^1 d\lambda \sum_{\alpha=1}^4 \zeta\xi_1^{k+\alpha}\langle \dot{t}_1^{\alpha+k}\dot{t}_1^{k+1}\rangle_{\zeta,\boldsymbol{\theta}}(\lambda)\right]^2\right\} \\ &= E^{1/2}\left\{\int_0^1\int_0^1 d\lambda\, d\lambda_1 \frac{\zeta^2}{N}\right. \\ &\qquad \left.\times \sum_{\alpha,\alpha_1=1}^4 |\langle \dot{t}_1^{\alpha+k}\dot{t}_1^{k+1}\rangle_{\zeta,\boldsymbol{\theta}(\lambda)}|\cdot|\langle \dot{t}_1^{\alpha_1+k}\dot{t}_1^{k+1}\rangle_{\zeta,\boldsymbol{\theta}(\lambda_1)}|\right\} \equiv Q \end{aligned}$$

Now the identity (3.17) and the bound (3.18) imply

$$Q \leqslant E^{1/2}\left\{\frac{\zeta^2}{N}\sum_{\alpha=1}^{4} \langle \dot{t}_1^{\alpha+k}\dot{t}_1^{k+1}\rangle_{\zeta,1}^2\right\}$$

$$\leqslant \text{const}\cdot E^{1/2}\left\{\frac{\zeta^2}{N}\sum_{\alpha=1}^{4} \langle(\dot{t}_1^{\alpha+k})^2\rangle_{\zeta,1}\langle(\dot{t}_1^{k+1})^2\rangle_{\zeta,1}\right\} \tag{3.20}$$

Inserting this inequality in the formula for C, we get

$$C \leqslant \text{const}\cdot E^{1/2}\left\{\frac{1}{N}\sum_{\alpha=1}^{4} \langle(\dot{t}_1^{\alpha+k})^2\rangle_{\zeta,1}\langle(\dot{t}_1^{k+1})^2\rangle_{\zeta,1}\right\}$$

$$\leqslant \text{const}\cdot E^{1/4}\left\{\left(\frac{1}{N}\right)^2\sum_{\alpha=1}^{4} \langle(\dot{t}_1^{\alpha+k})^2\rangle_{\zeta,1}^2\right\} E^{1/4}\left\{\langle(\dot{t}_1^{k})^2\rangle_{\zeta,1}^2\right\}$$

$$\leqslant \text{const}\cdot E^{1/2}\left\{\frac{1}{N}\langle(\dot{t}_1^{k})^2\rangle_{\zeta,1}^2\right\}$$

$$\leqslant \text{const}\cdot E^{1/2}\left\{\frac{1}{N^2}\sum_{\mu=1}^{p} \langle(\dot{t}_1^{\mu})^2\rangle_{\zeta,1}^2\right\} \tag{3.21}$$

The inequality (3.21) has been obtained by using repeatedly the symmetry of the expectation w.r.t. the indexes μ. By Lemma 3.1 we have that the quantity in the r.h.s. of (3.21) goes to zero. Thus Lemma 3.4 is proven.

In order to finish the proof of Lemma 2.2, we have to establish the following result:

Lemma 3.5:

$$\lim_{N\to\infty} E\left\{\left|\frac{1}{N}\sum_{\mu=1}^{p} \langle(\dot{t}^{\mu})^2\rangle_{\zeta,1} - \frac{1}{N}\sum_{\mu=1}^{p}\langle(\dot{t}^{\mu})^2\rangle\right|\right\} = 0$$

Proof. The s.a. of the free energy of the Hopfield model for fixed α and any value of β has been proven in ref. 6 using a martingale technique[6] analogous to that used in ref. 4. This property and the Bogolubov inequality[15] imply that

$$\lim_{N\to\infty} E\left\{\left|f\left(H_{N-1} + \frac{S_1}{\sqrt{N}}\sum_{\mu}\xi_1^{\mu}t_1^{\mu}\right) - f(H_N)\right|\right\} = 0 \tag{3.22}$$

Now we observe that $(1/N)\sum_{\mu=1}^{p}\langle(t^{\mu})^2\rangle_{\zeta,1}$ can be obtained as a linear combination of the first derivatives of the free energy with respect to β, $\varepsilon_{1,2}$, and ε^{μ}. Since the free energy is s.a., then its derivatives are also self-averaging[5] and we have the s.a. of $(1/N)\sum_{\mu=1}^{p}\langle (t^\mu)^2\rangle_{\zeta,1}$. From this fact we get that

$$\lim_{N\to\infty} E\left\{\left|\frac{1}{N}\sum_{\mu=1}^{p}\langle (t^\mu)^2\rangle_{\zeta,1}-\frac{1}{N}\sum_{\mu=1}^{p}\langle (t^\mu)^2\rangle\right|\right\}=0$$

However, we can write

$$\begin{aligned}
&E\left\{\left|\frac{1}{N}\sum_{\mu=1}^{p}\langle (t^\mu)^2\rangle_{\zeta,1}-\frac{1}{N}\sum_{\mu=1}^{p}\langle (t^\mu)^2\rangle\right|\right\}\\
&\quad\leqslant E\left\{\left|\frac{1}{N}\sum_{\mu=1}^{p}\langle (t^\mu)^2\rangle_{\zeta,1}-E\left\{\frac{1}{N}\sum_{\mu=1}^{p}\langle (t^\mu)^2\rangle_{\zeta,1}\right\}\right|\right\}\\
&\qquad+\left|E\left\{\frac{1}{N}\sum_{\mu=1}^{p}\langle (t^\mu)^2\rangle_{\zeta,1}\right\}-E\left\{\frac{1}{N}\sum_{\mu=1}^{p}\langle (t^\mu)^2\rangle\right\}\right|\\
&\qquad+E\left\{\left|\frac{1}{N}\sum_{\mu=1}^{p}\langle (t^\mu)^2\rangle-E\left\{\frac{1}{N}\sum_{\mu=1}^{p}\langle (t^\mu)^2\rangle\right\}\right|\right\}
\end{aligned}\tag{3.23}$$

The first and the third terms of (3.23) tend to zero as $N\to\infty$. In view of the above argument the second term in (3.23) also tends to zero because (3.22) implies that the two sequences of free energies have a common limit. Since the free energies are convex functions and their derivatives are continuous for almost every value of the parameters, the limits of the derivatives of the two sequences are also equal almost everywhere as a consequence of general properties of sequences of convex functions. Let us prove now the convergence to zero of

$$E\left\{\left|\frac{1}{N}\sum_{\mu}\langle t^\mu\rangle_{\zeta,1}^2-\frac{1}{N}\sum_{\mu}\langle t^\mu\rangle^2\right|\right\}\to 0$$

From (3.22) and the above arguments on the derivatives of the free energy it follows that

$$\begin{aligned}
&E\left\{\frac{1}{N}\sum_{\mu}\gamma^\mu\langle t^\mu\rangle_{\zeta,1}\right\}-E\left\{\frac{1}{N}\sum_{\mu}\gamma^\mu\langle t^\mu\rangle\right\}\\
&\quad=E\left\{\frac{1}{N}\left(\sum_{\mu}[\langle (t^\mu)^2\rangle_{\zeta,1}-\langle t^\mu\rangle_{\zeta,1}^2]-\sum_{\mu}[\langle (t^\mu)^2\rangle-\langle t^\mu\rangle^2]\right)\right\}
\end{aligned}\tag{3.24}$$

The l.h.s. goes to zero as $N\to\infty$ because it can be obtained as the derivative with respect to ε_2 of the free energy. Thus we get

$$E\left\{\frac{1}{N}\sum_{\mu}\langle t^\mu\rangle_{\zeta,1}^2-\frac{1}{N}\sum_{\mu}\langle t^\mu\rangle^2\right\}\to 0\tag{3.25}$$

If now we write the analog of the formula (3.23) for $\sum_\mu \langle t^\mu \rangle^2$ and use the same argument, we get finally that

$$E\left\{\left|\frac{1}{N}\sum_\mu \langle t^\mu \rangle^2_{\zeta,1} - \frac{1}{N}\sum_\mu \langle t^\mu \rangle^2\right|\right\} \to 0$$

If $(1/N)\sum_\mu \langle t^\mu \rangle^2_{\zeta,1}$ is s.a., Lemma 3.5 is proven, since this self-averaging property will be shown in the proof of Lemma 2.1.

Proof of Lemma 2.1:

(a) m^μ, $\mu = 1,..., p$, are s.a. because the free energy is s.a.,[6] and because these quantities can be obtained as derivatives of the free energy w.r.t. ε^μ and the derivation conserves the s.a. property.[5]

(b) Using again the property of s.a. of the free energy and its derivative w.r.t. ε_2, we have

$$\begin{aligned} A \equiv E\left\{\frac{1}{N}\sum_\nu \langle t^\nu \rangle^2_{\zeta,1} \left(\frac{1}{N}\sum_\mu \gamma^\mu \langle t^\mu \rangle_{\zeta,1}\right.\right. \\ \left.\left. - E\left\{\frac{1}{N}\sum_\mu \gamma^\mu \langle t^\mu \rangle_{\zeta,1}\right\}\right)\right\} \to 0 \end{aligned} \tag{3.26}$$

as $N \to \infty$. Integrating by parts with respect to γ^μ in (3.26), we obtain

$$\begin{aligned} A = \beta E &\left\{\frac{2}{N^2}\sum_{\mu,\nu} \langle t^\mu \rangle_{\zeta,1} \langle \dot{t}_1^\mu \dot{t}_1^\nu \rangle_{\zeta,1} \langle t^\nu \rangle_{\zeta,1}\right\} \\ &+ \beta E\left\{\frac{1}{N}\sum_\mu \langle t^\mu \rangle^2_{\zeta,1} \left[\frac{1}{N}\sum_\nu \langle (t^\nu)^2 \rangle_{\zeta,1}\right.\right. \\ &\left.\left. - E\left\{\frac{1}{N}\sum_\nu \langle (t^\nu)^2 \rangle_{\zeta,1}\right\}\right]\right\} \\ &- \beta E\left\{\left[\frac{1}{N}\sum_\mu \langle t^\mu \rangle^2_{\zeta,1} - E\left\{\frac{1}{N}\sum_\nu \langle (t^\nu) \rangle^2_{\zeta,1}\right\}\right]^2\right\} \end{aligned} \tag{3.27}$$

According to Lemma 3.5, $(1/N)\sum_\nu \langle (t^\nu)^2 \rangle_{\zeta,1}$ is s.a., so that the second term in the r.h.s. of (3.27) has zero limit as $N \to \infty$. So from (3.27) we obtain, as $N \to \infty$,

$$\begin{aligned} E&\left\{\left[\frac{1}{N}\sum_\mu \langle t^\mu \rangle^2_{\zeta,1} - E\left\{\frac{1}{N}\sum_\nu \langle (t^\nu) \rangle^2_{\zeta,1}\right\}\right]^2\right\} \\ &= 2E\left\{\frac{1}{N^2}\sum_{\mu,\nu} \langle t^\mu \rangle_{\zeta,1} \langle \dot{t}_1^\mu \dot{t}_1^\nu \rangle_{\zeta,1} \langle t^\nu \rangle_{\zeta,1}\right\} + o(1) \end{aligned} \tag{3.28}$$

But by the Schwarz inequality

$$E\left\{\frac{1}{N^2}\sum_{\mu,\nu}\langle t^\mu\rangle_{\zeta,1}\langle \dot{t}_1^\mu \dot{t}_1^\nu\rangle_{\zeta,1}\langle t^\nu\rangle_{\zeta,1}\right\}$$

$$\leqslant E^{1/2}\left\{\frac{1}{N^2}\sum_{\mu,\nu}\langle \dot{t}_1^\mu \dot{t}_1^\nu\rangle_{\zeta,1}^2\right\}E^{1/2}\left\{\frac{1}{N^2}\sum_{\mu,\nu}\langle t^\mu\rangle_{\zeta,1}^2\langle t^\nu\rangle_{\zeta,1}^2\right\}$$

$$\leqslant \text{const}\cdot E^{1/2}\left\{\frac{1}{N^2}\sum_{\mu,\nu}\langle \dot{t}_1^\mu \dot{t}_1^\nu\rangle_{\zeta,1}^2\right\} \tag{3.29}$$

Since according to Lemma 3.1 the r.h.s. in the last inequality has zero limit as $N\to\infty$, (3.28) and (3.29) complete the proof of Lemma 3.5 and also of Lemmas 2.1 and 2.2.

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